ON APPROXIMATION BY TRIGONOMETRIC SUMS AND POLYNOMIALS*

BY

DUNHAM JACKSON

The chief purpose of this paper, carried out in its second part, is the determination of numerical limits for certain constants, hitherto undetermined, which figure in the principal theorems of the first two parts (Abschnitte) of the author's thesis,† concerning the degree of approximation to a given continuous function $f(x)$ that can be attained uniformly in an interval by a polynomial of the nth degree in x, or for all values of x by a trigonometric sum of the nth order. By a " trigonometric sum of the nth order at most " is meant an expression of the form

$$a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx$$
$$+ b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx,$$

where the coefficients a_i, b_i, are constants. In the first part of the paper, occasion is taken to present considerably simplified proofs of those theorems themselves, with some modifications in statement.‡ The simplification depends on a recognition of the fact that it is possible to treat the trigonometric case directly and deduce thence the results in the polynomial case; the opposite order of treatment, employed in the thesis, requires a considerably longer discussion. The reader is referred to the introduction of the latter for a review of the literature of the subject, down to the spring of 1911. As far as logical development is concerned, the present paper may be read quite independently of the thesis, though, allowing for the fundamental change in arrangement, the first part of this article employs to a considerable extent the methods of the sections of the thesis referred to, and various matters of detail are more fully presented there than here.

In addition to the theorems thus based directly on the thesis, the article

* Presented to the Society, February 24 and April 27, 1912.

† *Über die Genauigkeit der Annäherung stetiger Funktionen durch ganze rationale Funktionen gegebenen Grades und trigonometrische Summen gegebener Ordnung.* Dissertation, Göttingen, 1911. Referred to hereafter as *Thesis*.

‡ See footnotes attached to the several theorems.

contains two others, the fifth and the tenth, the status of which is explained where they are taken up in detail. In connection with the latter, the answer is found to a question raised by FEJÉR* concerning the quantities, met with in the theory of Fourier's series, which he calls "LEBESGUE'S constants." Apart from this point, the substance of the article is indicated by the ten theorems, which are self-explanatory.

I. GENERAL THEOREMS

THEOREM I.† *There exists an absolute numerical constant K_1 having the following property: If $f(x)$ is a real function of the real variable x, of period 2π, which everywhere satisfies the Lipschitz condition*

$$(1) \qquad |f(x_2) - f(x_1)| \overline{\gtrless} \lambda |x_2 - x_1|,$$

λ being a constant, then there exists for every positive integral value of n a trigonometric sum $T_n(x)$, of the nth order at most, such that for all values of x

$$|f(x) - T_n(x)| \overline{\gtrless} \frac{K_1\lambda}{n}.$$

We suppose the function $f(x)$ given. We define an approximating function $I_m(x)$ as follows:

$$I_m(x) = h_m \int_{-\pi/2}^{\pi/2} f(x + 2u) \left[\frac{\sin mu}{m \sin u}\right]^4 du,$$

where m is any positive integer, and h_m is defined by the equation

$$(2) \qquad \frac{1}{h_m} = \int_{-\pi/2}^{\pi/2} \left[\frac{\sin mu}{m \sin u}\right]^4 du.$$

We note first that $I_m(x)$ is a trigonometric sum in x, of order $2(m-1)$ at most. For by the substitution $x + 2u = v$ we find

$$I_m(x) = \frac{h_m}{2} \int_{x-\pi}^{x+\pi} f(v) \left[\frac{\sin m\frac{v-x}{2}}{m \sin \frac{v-x}{2}}\right]^4 dv,$$

or, since $f(v)$ and the fourth power (in fact any even power) of the quantity in brackets are both unaltered by a change of 2π in the value of v,

*Since the completion of this paper, an article by GRONWALL, Mathematische Annalen, vol. 72 (1912), pp. 244–261, has come to hand, in which the same problem is solved.

† *Thesis*, Satz VI, and remark at the top of page 46. In the case of this theorem the proof here given is not essentially different from the old one.

$$I_m(x) = \frac{h_m}{2}\int_{-\pi}^{\pi} f(v)\left[\frac{\sin m\dfrac{v-x}{2}}{m\sin\dfrac{v-x}{2}}\right]^4 dv.$$

If we show that the fourth power in the integrand is a trigonometric sum in $(v-x)$, of order not higher than $2(m-1)$, it will follow at once that $I_m(x)$ is such a sum in x. Now if m is odd, setting $v-x=t$ for brevity, we have the well-known identity

$$\frac{\sin(mt/2)}{\sin(t/2)} = 2\left(\tfrac{1}{2} + \cos t + \cos 2t + \cdots + \cos\tfrac{1}{2}(m-1)t\right),$$

the right-hand member of which is a trigonometric sum of order $\frac{1}{2}(m-1)$ in t; its fourth power is then a trigonometric sum of order $2(m-1)$ in t. On the other hand, if m is even, $=2m'$, we may write

$$\frac{\sin(mt/2)}{\sin(t/2)} = \frac{\sin m't}{\sin t}\cdot\frac{\sin t}{\sin(t/2)}.$$

It is readily proved by induction that $\sin m't/\sin t$ is a trigonometric sum in t of order $m'-1=\frac{1}{2}(m-2)$; and while $\sin t/\sin(t/2) = 2\cos(t/2)$ itself is not a trigonometric sum in t, its square is such a sum, of order 1, and its fourth power is a sum of order 2. It follows that the function in the integrand is once more a trigonometric sum of order $2(m-1)$.

Having established this fact, we proceed to justify the use of the word "approximating" in connection with the function $I_m(x)$. Multiplying by $h_m f(x)$ the equation (2) which defines h_m, we obtain

$$f(x) = h_m\int_{-\pi/2}^{\pi/2} f(x)\left[\frac{\sin mu}{m\sin u}\right]^4 du.$$

Hence

$$I_m(x) - f(x) = h_m\int_{-\pi/2}^{\pi/2}[f(x+2u)-f(x)]\left[\frac{\sin mu}{m\sin u}\right]^4 du.$$

By the hypothesis expressed in (1),

$$|f(x+2u)-f(x)| \overline{\gtrless} 2\lambda|u|,$$

from which follows:

$$|I_m(x)-f(x)| \overline{\gtrless} 2\lambda h_m\int_{-\pi/2}^{\pi/2}|u|\left[\frac{\sin mu}{m\sin u}\right]^4 du = 4\lambda h_m\int_0^{\pi/2} u\left[\frac{\sin mu}{m\sin u}\right]^4 du,$$

or, if we substitute the value of h_m from (2) and make use of the fact that the

integrand there also is an even function,

$$(3)\qquad | I_m(x) - f(x) | \overline{\overline{<}} 2\lambda \frac{\displaystyle\int_0^{\pi/2} u \left[\frac{\sin mu}{m \sin u}\right]^4 du}{\displaystyle\int_0^{\pi/2} \left[\frac{\sin mu}{m \sin u}\right]^4 du}.$$

Let us consider first the denominator of this fraction.

$$\int_0^{\pi/2} \left[\frac{\sin mu}{m \sin u}\right]^4 du > \int_0^{\pi/2} \left[\frac{\sin mu}{mu}\right]^4 du$$
$$= \frac{1}{m}\int_0^{m\pi/2} \frac{\sin^4 u}{u^4}\, du \overline{\overline{>}} \frac{1}{m}\int_0^{\pi/2} \frac{\sin^4 u}{u^4}\, du,$$

and in this fourth member the integral factor is independent of m.

In the numerator, we make use of the fact that $(\sin u)/u \overline{\overline{>}} 2/\pi$ in the interval of integration, and see that

$$\int_0^{\pi/2} u \left[\frac{\sin mu}{m \sin u}\right]^4 du \overline{\overline{<}} \left(\frac{\pi}{2}\right)^4 \int_0^{\pi/2} u \left[\frac{\sin mu}{mu}\right]^4 du = \frac{1}{m^2}\left(\frac{\pi}{2}\right)^4 \int_0^{m\pi/2} \frac{\sin^4 u}{u^3}\, du$$
$$< \frac{1}{m^2}\left(\frac{\pi}{2}\right)^4 \int_0^{\infty} \frac{\sin^4 u}{u^3}\, du,$$

where m appears in the final expression only in the factor $1/m^2$.

By combination of these inequalities, it appears at once that the quantity $| I_m(x) - f(x) |$ does not exceed a constant multiple of λ/m, and Theorem I follows at once. To be sure, the order of the trigonometric sum $I_m(x)$ is not m, in general, but $2(m-1)$; and this takes on only even values, however m may be chosen. But the mere naming of these discrepancies suffices to show that they are inessential; if n is any positive integer, it will be possible to choose an m so that the function $I_m(x)$ will serve as the sum $T_n(x)$ required.*

Before going further, let us notice that in the application of Theorem I, or any similar theorem, if $f(x)$ is an even function, it may be assumed that the sums $T_n(x)$ have the same property. It is readily seen that the particular functions $I_m(x)$ used above will be even if $f(x)$ is even, but it is true more generally that if $f(x)$ is any even continuous function and $T_n(x)$ any trigonometric sum of the nth order whatever, and ϵ is a number such that

$$| f(x) - T_n(x) | \overline{\overline{<}} \epsilon,$$

* For greater detail on this point, see the remark preceding formula (17) in the proof of Theorem VI below.

for all values of x, then it is true also that

$$(4)\quad \left| f(x) - \frac{T_n(x)+T_n(-x)}{2} \right| = \left| \frac{f(x)-T_n(x)}{2} + \frac{f(-x)-T_n(-x)}{2} \right| \overline{\gtrless} \epsilon,$$

and $\frac{1}{2}(T_n(x) + T_n(-x))$ is obviously a trigonometric sum of the same order as $T_n(x)$ or lower, which is at the same time an even function.

This well-known fact being recalled, let us drop the assumption that $f(x)$ is periodic, and suppose merely that it satisfies the Lipschitz condition (1) in the closed interval $-1 \overline{\gtrless} x \overline{\gtrless} 1$. Then $g(x) = f(\cos x)$ is a function of x defined for all real values of x. Moreover, it satisfies everywhere the condition (1). For

$$|g(x_2) - g(x_1)| = |f(\cos x_2) - f(\cos x_1)| \overline{\gtrless} \lambda |\cos x_2 - \cos x_1| \overline{\gtrless} \lambda |x_2 - x_1|.$$

By Theorem I, then, it is possible to find for each value of n a trigonometric sum $T_n(x)$, of order n or less, such that the inequality

$$|g(x) - T_n(x)| \overline{\gtrless} \frac{K_1 \lambda}{n}$$

is everywhere satisfied. As $g(x)$ is a function of $\cos x$, it is an even function of x, and hence we may assume that $T_n(x)$ is an even function of x. Then $T_n(x)$ has no sine terms, but involves only cosines, and so is a polynomial in $\cos x$, of degree not higher than the nth,

$$T_n(x) = \Pi_n(\cos x).$$

We have thus a representation of $f(\cos x)$ by the polynomial $\Pi_n(\cos x)$, or of $f(x)$ by $\Pi_n(x)$, for all values of the argument in the interval $(-1, 1)$, with a maximum error not greater than $K_1 \lambda / n$.

If the function $f(x)$ were given in some other interval than $(-1, 1)$, we should reduce that interval to this by a linear transformation of the form $x' = Ax + B$. The transformed function would be represented by a polynomial of the same degree, with the same error, as the original one. The transformed function would still satisfy a Lipschitz condition, the coefficient λ in this condition being altered in the inverse ratio of the lengths of the intervals. For example, the problem of representing a function which satisfies a Lipschitz condition with coefficient λ in an interval of length 1 would be reduced to that of representing a function which satisfies a Lipschitz condition with coefficient $\frac{1}{2}\lambda$ in the interval $(-1, 1)$ of length 2. We are led to the general statement which follows:

THEOREM II.* *There exists an absolute numerical constant L_1 having the*

* *Thesis*, Satz IV. The concluding statement, as to the relative values of K_1 and L_1, was not contained in the thesis.

following property: If $f(x)$ is a function of x which satisfies the Lipschitz condition (1) *throughout a closed interval $a \leqq x \leqq b$, of length l, then there exists for every positive integral value of n a polynomial $\Pi_n(x)$, of degree n at most, such that, for all values of x in the interval,*

$$|f(x) - \Pi_n(x)| \leqq \frac{L_1 l \lambda}{n}.$$

In particular, the constant $L_1 = \frac{1}{2}K_1$ has this property, if K_1 is a constant possessing the property described in Theorem I.

A generalization of the first theorem is

THEOREM III.* *For each positive integral value of k there exists a constant K_k having the following property: If $f(x)$ is a function of period 2π possessing a $(k-1)$th derivative which everywhere satisfies the Lipschitz condition*

$$(5) \qquad |f^{(k-1)}(x_2) - f^{(k-1)}(x_1)| \leqq \lambda |x_2 - x_1|,$$

λ being a constant, then there exists for every positive integral value of n a trigonometric sum $T_n(x)$, of the nth order at most, such that for all values of x

$$|f(x) - T_n(x)| \leqq \frac{K_k \lambda}{n^k}.$$

Let us begin by considering the case $k = 2$. Instead of the previous definition of $I_m(x)$ we adopt the following:

$$I_m(x) = \frac{h_m}{2}\int_{-\pi/2}^{\pi/2}[f(x+2u) + f(x-2u)]\left[\frac{\sin mu}{m \sin u}\right]^4 du;$$

h_m has the same meaning as before. Various details of the earlier work will be again applicable, and need not be repeated explicitly. The new $I_m(x)$ is still a trigonometric sum in x, of order $2(m-1)$ or lower. We find

$$I_m(x) - f(x) = \frac{h_m}{2}\int_{-\pi/2}^{\pi/2}[f(x+2u) - 2f(x) + f(x-2u)]\left[\frac{\sin mu}{m \sin u}\right]^4 du.$$

By reason of the hypothesis concerning the derivative of $f(x)$,

$$f(x+2u) = f(x) + 2uf'(x + \theta_1 \cdot 2u) = f(x) + 2uf'(x) + \theta_1' \cdot 4\lambda u^2,$$

$$f(x-2u) = f(x) - 2uf'(x + \theta_2 \cdot 2u) = f(x) - 2uf'(x) + \theta_2' \cdot 4\lambda u^2,$$

where θ_1 and θ_2 are between 0 and 1, and θ_1' and θ_2' between -1 and 1. Hence

$$|f(x+2u) - 2f(x) + f(x-2u)| \leqq 8\lambda u^2,$$

* *Thesis*, Satz VII. The present statement is however more precise than the old one. See also footnote under Theorem IV.

and

$$(6)\quad | I_m(x) - f(x) | \overline{\lessgtr} 4\lambda h_m \int_{-\pi/2}^{\pi/2} u^2 \left[\frac{\sin mu}{m \sin u}\right]^4 du = 4\lambda \frac{\displaystyle\int_0^{\pi/2} u^2 \left[\frac{\sin mu}{m\sin u}\right]^4 du}{\displaystyle\int_0^{\pi/2} \left[\frac{\sin mu}{m \sin u}\right]^4 du}.$$

We have seen that the denominator of this fraction remains greater than a constant multiple of $1/m$. As to the numerator,

$$\int_0^{\pi/2} u^2 \left[\frac{\sin mu}{m\sin u}\right]^4 du < \left(\frac{\pi}{2}\right)^4 \int_0^{\pi/2} u^2 \left[\frac{\sin mu}{mu}\right]^4 du < \frac{1}{m^3}\left(\frac{\pi}{2}\right)^4 \int_0^{\infty} \frac{\sin^4 u}{u^2} du.$$

Hence the quantity $| I_m(x) - f(x) |$ does not exceed a constant multiple of λ / m^2, and Theorem II for $k = 2$ follows immediately.

In constructing a proof which shall be applicable for any value of k, we shall make use of an approximating function $I_m(x)$ which does not reduce to that just employed when $k = 2$, but to an analogous expression which might have been used instead of that one.

First, we show that if i and κ are positive integers the function

$$J(x) = \int_{-\pi/2}^{\pi/2} f(x + 2iu) \left[\frac{\sin mu}{m \sin u}\right]^{2\kappa} du$$

is a trigonometric sum in x. Let us make the substitution

$$x + 2iu = v.$$

The integral $J(x)$ becomes

$$\frac{1}{2i}\int_{x-i\pi}^{x+i\pi} f(v) \left[\frac{\sin m\dfrac{v-x}{2i}}{m \sin \dfrac{v-x}{2i}}\right]^{2\kappa} dv,$$

which is the same thing as

$$\frac{1}{2i}\int_{-i\pi}^{i\pi} f(v) \left[\frac{\sin m\dfrac{v-x}{2i}}{m \sin \dfrac{v-x}{2i}}\right]^{2\kappa} dv,$$

because of the periodicity of the integrand. From this form we see by a process of reasoning used before that $J(x)$ is at any rate a trigonometric sum in x/i. Let us break it up into i integrals, and write

$$2iJ(x) = \int_{-i\pi}^{-(i-2)\pi} + \int_{-(i-2)\pi}^{-(i-4)\pi} + \cdots + \int_{(i-2)\pi}^{i\pi}.$$

If we replace x by $x + 2\pi$ in the last of these integrals, we get

$$\int_{(i-2)\pi}^{i\pi} f(v) \left[\frac{\sin m \dfrac{v - (x + 2\pi)}{2i}}{m \sin \dfrac{v - (x + 2\pi)}{2i}} \right]^{2\kappa} dv,$$

and this, if we set $v - 2\pi = v'$ and remember that $f(v') = f(v)$, is equal to

$$\int_{(i-4)\pi}^{(i-2)\pi} f(v') \left[\frac{\sin m \dfrac{v' - x}{2i}}{m \sin \dfrac{v' - x}{2i}} \right]^{2\kappa} dv';$$

that is, the substitution of $x + 2\pi$ for x carries over the last of the i integrals into the next preceding one. Similarly, it will carry each of the others, except the first, into the one before it. The first will go over into

$$\int_{-(i+2)\pi}^{-i\pi} f(v') \left[\frac{\sin m \dfrac{v' - x}{2i}}{m \sin \dfrac{v' - x}{2i}} \right]^{2\kappa} dv' = \int_{(i-2)\pi}^{i\pi} f(v'') \left[\frac{\sin m \dfrac{v'' - x}{2i}}{m \sin \dfrac{v'' - x}{2i}} \right]^{2\kappa} dv'',$$

$$v'' = v' + 2i\pi;$$

the first integral goes over precisely into the ith. Hence $J(x)$ has the period 2π, and the terms in its expansion which involve multiples of x/i other than whole multiples of x must vanish.* It is of order $\kappa(m - 1)$ or less in x/i, accordingly of order not higher than $\kappa(m - 1)/i$ in x.

With this information, let us choose κ, for any given value of k, as the smallest integer such that $2\kappa - k > 1$, so that κ is perfectly determinate when k is given and set

$$I_m(x) = h_m \int_{-\pi/2}^{\pi/2} [\pm f(x + 2ku) \mp kf(x + 2(k-1)u) + \cdots$$
$$+ kf(x + 2u)] \left[\frac{\sin mu}{m \sin u} \right]^{2\kappa} du,$$

where the numerical coefficients of the terms in the first factor of the integrand are the binomial coefficients corresponding to the exponent k, and h_m is defined by the equation

$$\frac{1}{h_m} = \int_{-\pi/2}^{\pi/2} \left[\frac{\sin mu}{m \sin u} \right]^{2\kappa} du.$$

---

* This can be deduced without difficulty from the fact that a trigonometric sum can vanish identically only if all its coefficients are zero.

The function so defined is a trigonometric sum in x, of order $\kappa(m-1)$ at most.

The difference $I_m(x) - f(x)$ may be written in a form which differs from that of $I_m(x)$ only in having one more term, $-f(x)$, in the first factor of the integrand. If this factor is developed by Taylor's theorem, with the aid of (5), to terms of the kth degree in u, it is found that the terms of degree lower than the kth in the expansion combine so that their sum is identically zero, and the whole expression does not exceed a constant multiple of $\lambda|u|^k$ in absolute value. It will be seen on writing out the expansion at length that this statement will be justified if we can show that the function

$$S_{k,i}(t) = t^i - k(t+1)^i + \cdots + (-1)^{k-1}k(t+k-1)^i + (-1)^k(t+k)^i,$$

where the numerical coefficients are still the binomial coefficients for the exponent k, vanishes identically whenever $0 \overline{\gtrless} i \overline{\gtrless} k-1$, i being an integer; what is needed is only the fact that it vanishes when $t=0$. As

$$\frac{d}{dt}S_{k,i}(t) \equiv iS_{k,i-1}(t),$$

$S_{k,i-1}(t)$ vanishes identically if $S_{k,i}(t)$ does, and it is sufficient to prove for each value of k that $S_{k,k-1}(t)$ is identically zero. The desired proof is obtained by induction. Suppose $S_{k-1,k-2}(t) \equiv 0$; this is true, for example, if $k=2$. Then

$$\frac{d}{dt}S_{k-1,k-1}(t) \equiv (k-1)S_{k-1,k-2}(t) \equiv 0,$$

and hence $S_{k,k-1}(t)$, which is equal to $S_{k-1,k-1}(t) - S_{k-1,k-1}(t+1)$, is identically zero, and the induction is complete.* It follows that the absolute value of $I_m(x) - f(x)$ does not exceed

$$\lambda \frac{\displaystyle\int_0^{\pi/2} u^k \left[\frac{\sin mu}{m\sin u}\right]^{2\kappa} du}{\displaystyle\int_0^{\pi/2} \left[\frac{\sin mu}{m\sin u}\right]^{2\kappa} du}$$

multiplied by a quantity which is dependent on k and on nothing else. For the denominator and the numerator of this fraction we have the inequalities

$$\int_0^{\pi/2}\left[\frac{\sin mu}{m\sin u}\right]^{2\kappa} du > \int_0^{\pi/2}\left[\frac{\sin mu}{mu}\right]^{2\kappa} du \overline{\gtrless} \frac{1}{m}\int_0^{\pi/2}\frac{\sin^{2\kappa} u}{u^{2\kappa}}du;$$

$$\int_0^{\pi/} u^k\left[\frac{\sin mu}{m\sin u}\right]^{2\kappa} du < \left(\frac{\pi}{2}\right)^{2\kappa}\int_0^{\pi/2} u^k\left[\frac{\sin mu}{mu}\right]^{2\kappa} du < \frac{1}{m^{k+1}}\left(\frac{\pi}{2}\right)^{2\kappa}\int_0^{\infty}\frac{\sin^{2\kappa}u}{u^{2\kappa-k}}du;$$

* Cf. *Thesis*, pp. 29–30, where a somewhat different proof is given.

and the last integral actually exists, since we have supposed $2\kappa - k > 1$. Hence $| I_m(x) - f(x) |$ does not exceed a constant multiple of λ / m^k, and by establishing a suitable relation between m and n the functions $I_m(x)$ may be made to serve the purpose of the sums $T_n(x)$ demanded by Theorem III.

A corresponding generalization of Theorem II is possible. Restricting ourselves at first to the case $k = 2$, let us suppose that $f(x)$ is a function of x defined in the interval $-1 \leqq x \leqq 1$, and possessing there a derivative $f'(x)$ which satisfies the condition

$$|f'(x_2) - f'(x_1)| \leqq \lambda |x_2 - x_1|. \tag{7}$$

Then $g(x) = f(\cos x)$ is defined for all values of x; it has everywhere a derivative

$$g'(x) = -\sin x \cdot f'(\cos x),$$

and this satisfies the condition

$$\begin{aligned} | g'(x_2) - g'(x_1) | &= | -\sin x_2 f' \cos(x_2) + \sin x_1 f'(\cos x_1) | \\ &= | -\sin x_2 [f'(\cos x_2) - f'(\cos x_1)] - f'(\cos x_1)(\sin x_2 - \sin x_1) | \\ &\leqq |f'(\cos x_2) - f'(\cos x_1)| + |f'(\cos x_1)| \, |x_2 - x_1|. \end{aligned} \tag{8}$$

As we may subtract a linear function from $f(x)$ without altering the problem of approximating to it by a polynomial of given positive degree, we may assume without loss of generality that $f'(0) = 0$; then it follows from condition (7) that throughout the interval of definition $|f'(x)| \leqq \lambda$. Applying (7) once more in the relation (8), we find

$$| g'(x_2) - g'(x_1) | \leqq 2\lambda | x_2 - x_1 |.$$

Accordingly $g(x) = f(\cos x)$ can be approximately represented by a trigonometric sum of order n or less, and hence by a polynomial in $\cos x$ of degree n or less, for all values of x, with an error not exceeding $2K_2 \lambda / n^2$, where K_2 is the constant of Theorem III for the case $k = 2$. That is, $f(x)$ can be so represented by a polynomial of not higher degree than the nth in x, throughout the interval $-1 \leqq x \leqq 1$.

If $f(x)$ satisfies the condition (7) in some other interval than this, and the given interval is reduced to this one by a linear transformation of the form $x' = Ax + B$, the condition (7) will be preserved, except that the coefficient λ will be altered in the inverse ratio of the squares of the lengths of the intervals. If the given interval were of length 1, for example, the Lipschitz condition after the transformation to the interval $(-1, 1)$ would have the coefficient $\frac{1}{4}\lambda$, and a representation would be obtained with a maximum error not exceeding $\frac{1}{2}K_2 \lambda / n^2$.

The reasoning based on the hypothesis that $f(x)$ has a $(k-1)$th derivative satisfying (5) in a closed interval is similar. Suppose first that the interval is $(-1, 1)$. We consider again the function $g(x) = f(\cos x)$. The derivative $g^{(k-1)}(x)$ exists everywhere, and is given by a polynomial in the derivatives $f^{(i)}(\cos x)$, $i = 1, 2, \cdots, k-1$, and $\sin x$ and $\cos x$, which is linear in the derivatives of f. By subtracting a suitable polynomial of degree $k-1$ from $f(x)$ at the beginning, we can make

$$f^{(k-1)}(0) = f^{(k-2)}(0) = \cdots = f'(0) = 0,$$

so that each of these functions remains in absolute value less than or equal to λ throughout the interval, and each satisfies a Lipschitz condition with coefficient λ throughout the interval. The subtraction of this polynomial does not affect the degree of approximation to $f(x)$ which can be obtained by a polynomial of degree n, provided $n \gtreqqless k-1$, and we shall suppose that the subtraction has been performed. With this assumption, it appears from the expression for $g^{(k-1)}(x)$ that a relation holds of the form

$$| g^{(k-1)}(x_2) - g^{(k-1)}(x_1) | \lesseqqgtr A_k \lambda | x_2 - x_1 |,$$

where A_k is a constant dependent only on k. Hence $g(x) = f(\cos x)$ can be represented by a polynomial in $\cos x$ of degree n or lower, and $f(x)$ by such a polynomial in x, with a maximum error not exceeding $A_k K_k \lambda / n^k$.

To make the result applicable in the case of an arbitrary interval, we have merely to note that a linear transformation of the sort already considered, carrying one interval into another, changes the coefficient of the Lipschitz condition for $f^{(k-1)}(x)$ in the inverse ratio of the kth powers of the lengths of the intervals, without otherwise affecting the conditions of the problem.

A summary of the facts in this connection which we are now in a position to recognize may be made as follows:

THEOREM IV.* *For each positive integral value of k there exists a constant L_k having the following property: If $f(x)$ is a function of x possessing a $(k-1)$th derivative which satisfies the Lipschitz condition (5) throughout an interval $a \lesseqqgtr x \lesseqqgtr b$, of length l, then there exists for every positive integral value of n,*

* *Thesis*, Satz IV*a*. The restriction $n \geqq k-1$ was overlooked in the thesis; in the process of reasoning, the details of which were dismissed with the words "indem wir wie vorhin die Bedeutung der verschiedenen Grössen überlegen" (p. 39), it was tacitly assumed that $|f(x)|$ itself remains inferior to a constant multiple of λ, which would be true in general only after subtraction of a polynomial of the $(k-1)$th degree. That the theorem is absurd without this restriction is shown by the example $f(x) \equiv x^2$, $k = 3$, $\lambda = 0$, $n = 1$. The restriction does not apply to Theorem III; and the question does not arise with reference to the less definite statement of Theorems II and VII of the thesis. As to the last two propositions, which are contained in IV and III respectively of the present paper, it is seen by comparing the old treatment and the new one that either may be proved independently and used to establish the other.

greater than or equal to $k-1$, *a polynomial* $\Pi_n(x)$, *of the nth degree at most, such that, for all values of* x *in the interval,*

$$|f(x) - \Pi_n(x)| \leqq \frac{L_k l^k \lambda}{n^k}.$$

In particular, in the case $k = 2$, *the constant* $L_2 = \frac{1}{2}K_2$ *has this property, if* K_2 *is a corresponding constant for Theorem III.*

It has frequently been noted* that if $f(x)$ is an integrable function of period 2π, and there exists a trigonometric sum $T_n(x)$, of order n or lower, $n \geqq 2$, such that

$$|f(x) - T_n(x)| \leqq \epsilon$$

for all values of x, and if $S_n(x)$ is the partial sum to terms of the nth order of the formal development of $f(x)$ in Fourier's series, then

$$|f(x) - S_n(x)| \leqq K\epsilon \log n,$$

for all values of x, where K is an absolute numerical constant. The argument is, briefly, as follows: Let $f(x)$ be expressed as $T_n(x) + [f(x) - T_n(x)]$. The partial sum of the Fourier's series for $T_n(x)$ is $T_n(x)$ itself, so that an error arises only in the representation of $f(x) - T_n(x)$. As this function never exceeds ϵ in absolute value, its partial sum does not exceed $\epsilon\rho_n$, where

$$(9) \qquad \rho_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \left| \frac{\sin(2n+1)\left(\frac{u-x}{2}\right)}{2\sin\left(\frac{u-x}{2}\right)} \right| du = \frac{2}{\pi}\int_0^{\pi/2} \frac{|\sin(2n+1)t|}{\sin t}\, dt,$$

and the difference between the partial sum and the function can not exceed $\epsilon(1+\rho_n)$. Finally, it is found that, when $n > 1$, ρ_n does not exceed a certain multiple† of $\log n$. This fact, in conjunction with Theorems I and III (of which the first may be regarded as a special case of the second), establishes the following result:

THEOREM V.‡ *If* $f(x)$ *is a function of period* 2π *possessing a* $(k-1)$*th*

* See, for example: LEBESGUE, Annales de la Faculté de Toulouse, series 3, vol. 1 (1909), pp. 25–117; pp. 116–117; LEBESGUE, Bulletin de la Soc. Math. de France, vol. 38 (1910), pp. 184–210 (referred to hereafter as LEBESGUE II); pp. 196–197, 201, of the volume, 13–14, 18, of the article. Also, *Thesis*, pp. 49–51.

† For a proof of this well-known fact, see the concluding paragraphs of the present paper.

‡ A somewhat less precise theorem was indicated, but not formally stated, in a recent paper of the author in these Transactions, vol. 13 (1912), pp. 305–318, in a remark following Theorem II of that paper, which is an analogous theorem concerning Legendre's series. The case $k = 1$ of the present theorem is treated by a different method in the article LEBESGUE II, pp. 199–201 of the volume, 16–18 of the article. The case $k = 2$, less precisely stated, is closely parallel to a statement of WEYL, Rendiconti del Circolo Matematico di Palermo, vol. 32 (1911), pp. 118–131; p. 128, footnote; but neither includes the other.

derivative which satisfies the Lipschitz condition (5), *and* K *and* K_k *are the numerical constants already so designated, then* $f(x)$ *is everywhere approximately represented by the partial sum to terms of the nth order,* $n > 1$, *of its development in Fourier's series, with an error not exceeding*

$$\frac{KK_k \lambda \log n}{n^k}.$$

II. Numerical Determinations

We turn our attention now to the task of finding numerical values for some of the constants previously left undetermined. Confining ourselves for the present to the case discussed in Theorem I, we suppose a periodic function $f(x)$ given which satisfies the condition (1) for all values of the variables x_1 and x_2, and construct a function $I_m(x)$, which is a trigonometric sum in x of order $2(m-1)$ at most, such that the relation (3) is satisfied for all values of x. We have seen that for positive integral values of m the denominator of the fraction is never less than a certain constant multiple of $1/m$, and the numerator does not exceed a constant multiple of $1/m^2$. We shall undertake to discover something as to the magnitude of these constant multipliers.

Let us set

$$J_m = m \int_0^{\pi/2} \left[\frac{\sin mu}{m \sin u}\right]^4 du.$$

Making use of the identities

$$\left[\frac{\sin 2u}{\sin u}\right]^4 = 2 \cos 4u + 8 \cos 2u + 6,$$

$$\left[\frac{\sin 3u}{\sin u}\right]^4 = 2 \cos 8u + 8 \cos 6u + 20 \cos 4u + 32 \cos 2u + 19, \tag{10}$$

$$\left[\frac{\sin 4u}{\sin u}\right]^4 = 2 \cos 12u + 8 \cos 10u + 20 \cos 8u + 40 \cos 6u + 62 \cos 4u + 80 \cos 2u + 44,$$

we find the values

$$J_1 = \frac{\pi}{2} = .500\pi, \qquad J_2 = \frac{3\pi}{8} = .375\pi,$$

$$J_3 = \frac{19\pi}{54} = .352\pi -, \qquad J_4 = \frac{11\pi}{32} = .344\pi - .$$

On the other hand, we shall show that

$$\lim_{m=\infty} J_m = \int_0^\infty \frac{\sin^4 u}{u^4} du,$$

and that the value of the expression on the right is $\pi/3$.

To prove the first of these statements, we make a change of variable in the expression defining J_m, and write

$$J_m = \int_0^{m\pi/2} \frac{\sin^4 u}{u^4}\, du + \int_0^{m\pi/2} \sin^4 u \left(\left[m \sin \frac{u}{m} \right]^{-4} - u^{-4} \right) du$$
$$= J_{m1} + J_{m2}.$$

We note for reference the fact, of which we shall make repeated use, that

(11) $$m \sin \frac{u}{m} = u \left(\sin \frac{u}{m} \Big/ \frac{u}{m} \right),$$

and accordingly, if we restrict u to the interval $(0, m\pi/2)$,

(12) $$u \geqq m \sin \frac{u}{m} \geqq \frac{2}{\pi} u.$$

We proceed to estimate the magnitude of the term J_{m2} above. Assuming that u belongs to the interval of integration, we find

$$m \sin \frac{u}{m} = u \left[1 - \frac{1}{6} \frac{u^2}{m^2} + \cdots \right],$$

$$\left[m \sin \frac{u}{m} \right]^4 = u^4 \left[1 - \frac{4}{6} \frac{u^2}{m^2} + \cdots \right],$$

$$u^4 - \left[m \sin \frac{u}{m} \right]^4 = \frac{4}{6} \frac{u^6}{m^2} + \cdots < a \frac{u^6}{m^2},$$

where a is an absolute constant. Applying this inequality and (12), we have

$$\left[m \sin \frac{u}{m} \right]^{-4} - u^{-4} = \frac{u^4 - m^4 \sin^4 (u/m)}{m^4 u^4 \sin^4 (u/m)} < \left(\frac{\pi}{2} \right)^4 \frac{a}{m^2 u^2} = \frac{a_1}{m^2 u^2},$$

where $a_1 = (\pi/2)^4 a$; and

$$J_{m2} < \frac{a_1}{m^2} \int_0^{m\pi/2} \frac{\sin^4 u}{u^2}\, du.$$

Consequently

$$\lim_{m=\infty} J_{m2} = 0,$$

and

$$\lim_{m=\infty} J_m = \lim_{m=\infty} J_{m1} = \int_0^\infty \frac{\sin^4 u}{u^4}\, du.$$

The value of this expression may be determined by integration by parts:

$$\int_0^\infty \frac{\sin^4 u}{u^4}\, du = \frac{4}{3} \int_0^\infty \frac{\sin^3 u \cos u}{u^3}\, du = \frac{2}{3} \int_0^\infty \frac{3 \sin^2 u \cos^2 u - \sin^4 u}{u^2}\, du$$

$$= \frac{2}{3}\int_0^\infty \frac{3\sin^2 u - 4\sin^4 u}{u^2}\,du;$$

$$\int_0^\infty \frac{\sin^2 u}{u^2}\,du = \frac{\pi}{2},$$

$$(13)\quad \int_0^\infty \frac{\sin^4 u}{u^2}\,du = 4\int_0^\infty \frac{\sin^3 u\cos u}{u}\,du = \int_0^\infty \frac{\sin 2u - \frac{1}{2}\sin 4u}{u}\,du$$

$$= \int_0^\infty \frac{\sin u}{u}\,du - \frac{1}{2}\int_0^\infty \frac{\sin u}{u}\,du = \frac{\pi}{4}.$$

Hence

$$\lim_{m=\infty} J_m = \int_0^\infty \frac{\sin^4 u}{u^4}\,du = \frac{2}{3}\left(\frac{3\pi}{2} - \pi\right) = \frac{\pi}{3}.$$

We are led to inquire whether it is universally true that each value of J_m is greater than the following one; it turns out that the answer is in the affirmative. In carrying through the general proof we shall assume that $m \overline{\gtrless} 4$, as we already know that the statement is true for smaller values of m. We may write

$$J_{m+1} - J_m = \int_0^{(m+1)\pi/2}\left[\frac{\sin u}{(m+1)\sin\dfrac{u}{m+1}}\right]^4 du - \int_0^{m\pi/2}\left[\frac{\sin u}{m\sin\dfrac{u}{m}}\right]^4 du$$

$$= \int_{m\pi/2}^{(m+1)\pi/2}\left[\frac{\sin u}{(m+1)\sin\dfrac{u}{m+1}}\right]^4 du$$

$$+ \int_0^{m\pi/2}\sin^4 u\left(\left[(m+1)\sin\frac{u}{m+1}\right]^{-4} - \left[m\sin\frac{u}{m}\right]^{-4}\right)du$$

$$= H_{m1} + H_{m2}.$$

The first of these terms is positive. The second is negative, since the right-hand side of (11) increases as m increases and u/m decreases. We shall show that the negative term is numerically the larger.

Taking first the term H_{m1}, we have within the interval of integration, for $m \overline{\gtrless} 4$,

$$\frac{4}{5}\cdot\frac{\pi}{2} \overline{\lessgtr} \frac{m}{m+1}\cdot\frac{\pi}{2} \overline{\lessgtr} \frac{u}{m+1} \overline{\lessgtr} \frac{\pi}{2},$$

$$\sin\frac{u}{m+1} \overline{\gtrless} \sin\frac{4\pi}{10} > .951,$$

$$1\Big/\sin^4\frac{u}{m+1} < 1.23.$$

By applying this inequality, and then the identity

$$\sin^4 u = \tfrac{1}{8}\cos 4u - \tfrac{1}{2}\cos 2u + \tfrac{3}{8},$$

we obtain the relation

$$H_{m1} < 1.23 \int_{m\pi/2}^{(m+1)\pi/2} \frac{\sin^4 u}{(m+1)^4} du = 1.23 \cdot \frac{3\pi}{16} \cdot \frac{1}{(m+1)^4} < \frac{.73}{(m+1)^4}.$$

To form an estimate of the magnitude of the other integral H_{m2}, we need to examine somewhat carefully the behavior of the function $(\sin x)/x$ in the interval $(0, \pi/2)$. If x_1 and x_2 are any two values of the variable,

$$\frac{\sin x_2}{x_2} - \frac{\sin x_1}{x_1} = (x_2 - x_1)\left[\frac{d}{dx}\frac{\sin x}{x}\right]_{x=\xi},$$

where ξ is some value in the interval (x_1, x_2). Now

$$\frac{d}{dx}\frac{\sin x}{x} = \frac{x\cos x - \sin x}{x^2}.$$

In the interval $0 \gtreqless x \gtreqless \pi/2$,

$$\cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24} < 1 - .397\, x^2,$$

$$x\cos x < x - .397\, x^3,$$

$$\sin x > x - \frac{x^3}{6} > x - .167\, x^3,$$

$$x\cos x - \sin x < -\, .230\, x^3,$$

$$\frac{d}{dx}\frac{\sin x}{x} < -\, .230 x.$$

If x_1 is less than x_2, then $-\,\xi$ is less than $-\,x_1$, and hence

$$\frac{\sin x_2}{x_2} - \frac{\sin x_1}{x_1} < -\, .230\, x_1 (x_2 - x_1).$$

In the interval of integration, u/m and $u/(m+1)$ belong to the interval $(0, \pi/2)$, accordingly

$$m\sin\frac{u}{m} - (m+1)\sin\frac{u}{m+1} = u\left(\frac{\sin(u/m)}{u/m} - \frac{\sin(u/(m+1))}{u/(m+1)}\right)$$

$$< u\left(-\, .230\frac{u}{m+1}\right)\left(\frac{u}{m} - \frac{u}{m+1}\right) \tag{14}$$

$$= -\, .230\frac{u^3}{m(m+1)^2}.$$

This may be written

$$1 - (m + 1) \sin \frac{u}{m+1} \bigg/ \left(m \sin \frac{u}{m} \right) < - .230 \frac{u^3}{m(m+1)^2} \bigg/ \left(m \sin \frac{u}{m} \right).$$

Setting the left-hand side of this relation equal to $1 - t$, for the moment, we note that

$$1 - t^4 = (1 - t)(1 + t + t^2 + t^3) < 4(1 - t),$$

since $t > 1$ and $1 - t$ is negative, or, in the old notation,

$$1 - \left[(m+1) \sin \frac{u}{m+1} \bigg/ \left(m \sin \frac{u}{m} \right) \right]^4 < - .920 \frac{u^3}{m(m+1)^2} \bigg/ \left(m \sin \frac{u}{m} \right),$$

$$\left[m \sin \frac{u}{m} \right]^4 - \left[(m+1) \sin \frac{u}{m+1} \right]^4 < - .920 \frac{u^3}{m(m+1)^2} \left[m \sin \frac{u}{m} \right]^3.$$

Hence

$$\left[(m+1) \sin \frac{u}{m+1} \right]^{-4} - \left[m \sin \frac{u}{m} \right]^{-4} < \frac{- .920 \dfrac{u^3}{m(m+1)^2}}{m \sin \dfrac{u}{m} \left[(m+1) \sin \dfrac{u}{m+1} \right]^4}.$$

Strengthening this inequality still further by means of (12), we recognize finally that

$$H_{m2} < \frac{- .920}{m(m+1)^2} \int_0^{m\pi/2} \frac{\sin^4 u}{u^2} du.$$

We have already found (see (13) above) that

$$\int_0^{\infty} \frac{\sin^4 u}{u^2} du = \frac{\pi}{4},$$

so that with the assumption that $m \overline{\gtrless} 4$,

$$\int_0^{m\pi/2} \frac{\sin^4 u}{u^2} du \overline{\gtrless} \int_0^{2\pi} \frac{\sin^4 u}{u^2} du = \frac{\pi}{4} - \int_{2\pi}^{\infty} \frac{\sin^4 u}{u^2} du > \frac{\pi}{4} - \int_{2\pi}^{\infty} \frac{du}{u^2} > .625,$$

and

$$H_{m2} < \frac{- .575}{m(m+1)^2} < \frac{- .575}{(m+1)^3} < \frac{- 2.8}{(m+1)^4}.$$

Comparing this with the upper limit which was found for the magnitude of H_{m1}, we see that

$$J_{m+1} - J_m = H_{m1} + H_{m2} < 0.$$

Since J_m is thus shown to decrease as m increases, and $\lim_{m\infty} J_m = \pi/3$, we have for all values of m

$$J_m > \frac{\pi}{3}.$$

Now let

$$J'_m = m^2 \int_0^{\pi/2} u \left[\frac{\sin mu}{m \sin u}\right]^4 du .$$

Using once more the identities (10), we find

$$(15)\qquad \begin{aligned} J'_1 &= 1.234 - , & J'_2 &= .851 - , \\ J'_3 &= .778 - , & J'_4 &= .745 - . \end{aligned}$$

We shall not go so far as to determine whether J'_m always decreases as m increases, but we shall show that, for $m > 4$, $J'_m < .759$.

Following the same procedure as in the case of J_m, we write

$$\begin{aligned} J'_{m+1} - J'_m &= \int_{m\pi/2}^{(m+1)\pi/2} u \left[\frac{\sin u}{(m+1)\sin \dfrac{u}{m+1}}\right]^4 du \\ &\quad + \int_0^{m\pi/2} u \sin^4 u \left(\left[(m+1)\sin\frac{u}{m+1}\right]^{-4} - \left[m \sin\frac{u}{m}\right]^{-4}\right) du \\ &= H'_{m1} + H'_{m2} . \end{aligned}$$

For the first term, we find

$$\begin{aligned} H'_{m1} &< 1.23 \int_{m\pi/2}^{(m+1)\pi/2} \frac{u \sin^4 u}{(m+1)^4} du \\ &< 1.23 \int_{m\pi/2}^{(m+1)\pi/2} \frac{(m+1)\pi}{2} \cdot \frac{\sin^4 u}{(m+1)^4} du < \frac{1.15}{(m+1)^3} . \end{aligned}$$

In the second term, supposing $m \gtreqless 4$, we apply again most of the work done in connection with H_{m2}, and find

$$H'_{m2} < \frac{-.920}{m(m+1)^2} \int_0^{m\pi/2} \frac{\sin^4 u}{u} du \lesseqgtr \frac{-.920}{m(m+1)^2} \int_0^{2\pi} \frac{\sin^4 u}{u} du .$$

For a rough estimate of the value of the last integral, we find

$$\begin{aligned} \int_0^{2\pi} \frac{\sin^4 u}{u} du &= \int_0^{\pi/2} + \int_{\pi/2}^{\pi} + \int_{\pi}^{3\pi/2} + \int_{3\pi/2}^{2\pi} \\ &> \frac{2}{\pi}\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4}\right) \int_0^{\pi/2} \sin^4 u \, du = \frac{25}{32} , \end{aligned}$$

and so

$$H'_{m2} < \frac{-.71}{m(m+1)^2} < \frac{-.71}{(m+1)^3} .$$

This inequality, combined with the one previously obtained for H'_{m1}, gives

$$J'_{m+1} - J'_m < \frac{.44}{(m+1)^3} .$$

Repeated application of this relation leads to the result

$$J'_m - J'_4 < .44\left(\frac{1}{5^3} + \frac{1}{6^3} + \cdots + \frac{1}{(m+1)^3}\right) < .44\int_4^\infty \frac{du}{u^3} < .014;$$

and hence, as we have seen that $J'_4 < .745$, it follows that

$$(16) \qquad J'_m < .759$$

for all values of m after the first three.

The results thus obtained give to the inequality (3), for $m \geqq 4$, the form

$$| I_m(x) - f(x) | \leqq 2\lambda \frac{.759}{m^2} \cdot \frac{m}{\pi / 3} \leqq \frac{1.45\lambda}{m}.$$

We still have to take account of the circumstance that the index m is not the order of the trigonometric sum $I_m(x)$, which we merely know to be not greater than $2(m-1)$. If n is any positive integer, let m be the integer such that

$$2(m-1) \leqq n < 2m,$$

and let us set

$$I_m(x) = T_n(x).$$

Then $T_n(x)$ is a trigonometric sum of the nth order at most, and as $1/m < 2/n$, we have for $m \geqq 4$, $n \geqq 6$,

$$(17) \qquad |f(x) - T_n(x)| \leqq \frac{2.90\lambda}{n}.$$

As for the first five values of n, to which the corresponding values of m are $1, 2, 2, 3, 3$ respectively, n is so much less than $2m$ in these cases that the relation (17) may still be obtained as a direct consequence of (15), although (16) no longer holds.

It is obvious that with a little more attention to details it would be possible to replace the constant 2.90 by a somewhat smaller value, even while we keep the same approximation-functions $T_n(x)$; and we have no reason to suppose that these are the best that can be obtained. It is easy, however, to set a limit below which the constant can not be reduced. Suppose that $f(x)$ is the function of period 2π which takes on the values $(-1)^p \pi/(2n+2)$ at the points $2p\pi/(2n+2)$, $p = 0, \pm 1, \pm 2, \cdots$, and is linear in each interval between two successive points of this set. Then $f(x)$ satisfies the Lipschitz condition (1) with $\lambda = 1$. As it is furthermore an even function, we may, in showing that every trigonometric sum of order n or lower must differ from $f(x)$ at some point or other by at least a certain amount, restrict ourselves at the outset to sums of cosines, in consequence of the line of reasoning that was based on the formulæ (4) above. In order to represent $f(x)$ with

an error always less than $\pi/(2n+2)$, a sum $T_n(x)$ must take on positive and negative values alternately at the $n+2$ points $2p\pi/(2n+2)$, $p=0, 1, \cdots, n+1$, and so must vanish at $n+1$ interior points of the interval $(0, \pi)$, — for $n+1$ different values of $\cos x$. As this is impossible for a cosine-sum of the nth order, which is a polynomial of the nth degree in $\cos x$, every trigonometric sum of the nth order or less must differ from $f(x)$ at some point by at least $\pi/(2n+2)$ or *

$$\frac{n}{n+1}\cdot\frac{\pi/2}{n}.$$

The factor $n/(n+1)$ approaches unity when $n=\infty$, and hence the statement of Theorem I surely becomes false if a numerical value smaller than $\pi/2$ is put in place of K_1. Just what is the smallest correct value of K_1 remains undetermined within the limits $(\pi/2, 2.90)$. In view of the fact that the decimal places of the latter number do not even signify a limitation of the method used, but only of the extent to which the calculation was pushed, we may perhaps best sum up our knowledge as follows:

THEOREM VI. *The statement of Theorem I remains correct if the undetermined constant K_1 is replaced by the number* 3 (*or even by a somewhat smaller number*), *but not if it is replaced by a numerical value smaller than $\pi/2$.*

We have already seen that the constant L_2 of Theorem II may be taken half as large as K_1. On the other hand, it is easy to show by the construction of particular functions $f(x)$ closely analogous to those introduced just above that L_1 can not be smaller than $\frac{1}{2}$. These facts may be put together as

THEOREM VII. *The constant L_2 of Theorem II may be replaced by the numerical value $\frac{3}{2}$, but not by a value smaller than $\frac{1}{2}$.*

To go one step further, suppose now that the periodic function $f(x)$ has a first derivative $f'(x)$ which everywhere satisfies the condition (7). We have seen how to define a function $I_m(x)$ which is a trigonometric sum of order $2(m-1)$ at most, and which satisfies the relation (6).

Let

$$J''_m = m^3\int_0^{\pi/2} u^2\left[\frac{\sin mu}{m\sin u}\right]^4 du.$$

The first few values of this quantity are

$$\begin{aligned} &J''_1 = 1.292-, \qquad J''_2 = .931-,\\ &J''_3 = .914-, \qquad J''_4 = .868-. \end{aligned} \tag{18}$$

* For a general theorem of which we are here using a very special case, see L. TONELLI, Annali di Matematica, series 3, vol. 15 (1908), pp. 47–119; p. 103; and, for the polynomial case, KIRCHBERGER, *Ueber Tchebychefsche Annäherungsmethoden*, Dissertation, Göttingen, 1902, p. 16.

We shall content ourselves with showing that for larger values of m

$$J''_m < 1.19.$$

We assume $m \geqq 4$, and write

$$J''_{m+1} - J''_m = \int_{m\pi/2}^{(m+1)\pi/2} u^2 \left[\frac{\sin u}{(m+1)\sin \dfrac{u}{m+1}} \right]^4 du$$

$$+ \int_0^{m\pi/2} u^2 \sin^4 u \left(\left[(m+1) \sin \frac{u}{m+1} \right]^{-4} - \left[m \sin \frac{u}{m} \right]^{-4} \right) du$$

$$= H''_{m1} + H''_{m2}.$$

For the first term,

$$H''_{m1} < 1.23 \int_{m\pi/2}^{(m+1)\pi/2} \frac{u^2 \sin^4 u}{(m+1)^4} du$$

$$< 1.23 \int_{m\pi/2}^{(m+1)\pi/2} \frac{(m+1)^2 \pi^2}{2^2} \cdot \frac{\sin^4 u}{(m+1)^4} du < \frac{1.80}{(m+1)^2}.$$

For the second term,

$$H''_{m2} < \frac{-.920}{m(m+1)^2} \int_0^{m\pi/2} \sin^4 u \, du \leqq \frac{-.920}{m(m+1)^2} \cdot \frac{3m\pi}{16} < \frac{-.54}{(m+1)^2}.$$

Combining the two, we find

$$J''_{m+1} - J''_m < \frac{1.26}{(m+1)^2},$$

when $m \geqq 4$. Repeated application of this inequality gives the result

$$J''_m < J''_4 + 1.26 \int_4^\infty \frac{du}{u^2} < .87 + .32 = 1.19,$$

which is true also when $m = 2$ or 3, by (18).

It follows that

$$| I_m(x) - f(x) | < 4\lambda \frac{1.19}{m^3} \cdot \frac{m}{\pi/3} \leqq \frac{4.55\lambda}{m^2},$$

when $m \geqq 2$. If n is any positive integer, we associate with it a suitable m, by the same convention as in the proof of Theorem VI, and obtain a trigonometric sum $T_n(x)$ of order n or lower, of which it can be affirmed, as $1/m^2 < 4/n^2$ in general and $1/m^2 = 1/n^2$ for $n = 1$, that

$$|f(x) - T_n(x)| \leqq \frac{18.2\lambda}{n^2}.$$

We may state, then, with the understanding that further advance along the same line is surely possible,

THEOREM VIII. *The constant K_2 of Theorem III may be given the value* 20.

Furthermore, since L_2 in Theorem IV may be taken half as large as K_2, we have

THEOREM IX. *The constant L_2 of Theorem IV may be given the value* 10.

In conclusion, we come back to the subject of the constants ρ_n, referred to in connection with Theorem V. Lebesgue* has obtained inequalities that are satisfied by these numbers when n is sufficiently large, and Fejér† has deduced an asymptotic formula which represents them approximately for large values of n. But these relations ‡ do not immediately yield numerical information about the values of ρ_n for specific values of n. It is our next purpose to gain such information, though the results obtained will not be of a high degree of refinement.

If we set

$$j_m = \int_0^{\pi/2} \frac{|\sin mt|}{\sin t}\, dt,$$

then the constants in question, according to (9), are defined by the equation

$$\rho_n = \frac{2}{\pi} j_{2n+1}.$$

Actual integration, with the aid of the identities

$$\frac{\sin 3t}{\sin t} = 1 + 2\cos 2t, \qquad \frac{\sin 5t}{\sin t} = 1 + 2\cos 2t + 2\cos 4t,$$

gives the values

$$j_1 = \frac{\pi}{2} = 1.58-, \qquad j_3 = 2.26-, \qquad j_5 = 2.58-. \tag{19}$$

Transforming the integrals as we have done in similar cases before, let us set

$$j_{m+1} - j_m = \int_{m\pi/2}^{(m+1)\pi/2} \frac{|\sin t|}{(m+1)\sin\dfrac{t}{m+1}}\, dt$$

* LEBESGUE II, pp. 196–198 of the volume, 13–15 of the article.

† FEJÉR, Crelle's Journal, vol. 138 (1910), pp. 22–53; p. 30.

‡ In his recent paper, already referred to in a footnote, GRONWALL has elaborated the theory of these constants to a much greater extent than is attempted here. He obtains the simple relation

$$\rho_n = \frac{4}{\pi^2}\log n + h_n, \qquad \frac{4}{\pi^2} + \frac{9}{4} > h_n > \frac{4C}{\pi^2},$$

C being Euler's constant; this gives a sharper inequality than (20) for large but not for small values of n. He gives also an expansion of ρ_n according to descending powers of $2n+1$ (apart from the logarithmic term), with an estimate of the magnitude of the remainder.

$$+ \int_0^{m\pi/2} |\sin t| \left(\left[(m+1) \sin \frac{t}{m+1} \right]^{-1} - \left[m \sin \frac{t}{m} \right]^{-1} \right) dt$$

$$= h_{m1} + h_{m2}.$$

In the first term, if we assume $m \geqq 5$, as we shall do from now on, $\sin \dfrac{t}{m+1}$ is between $\sin 75° > .965$ and 1, accordingly

$$h_{m1} < \frac{1}{.965} \int_{m\pi/2}^{(m+1)\pi/2} \frac{|\sin t|}{m+1} dt < \frac{1.04}{m+1}.$$

Referring to (14), and using (12) once more, we have the inequality

$$\left[(m+1) \sin \frac{t}{m+1} \right]^{-1} - \left[m \sin \frac{t}{m} \right]^{-1} < -.230 \frac{t}{m(m+1)^2},$$

from which follows

$$h_{m2} < \frac{-.230}{m(m+1)^2} \int_0^{m\pi/2} t\, |\sin t|\, dt$$

$$< \frac{-.230}{m(m+1)^2} \big(1 + 2 + \cdots + (m-1)\big) \frac{\pi}{2} \int_0^{\pi/2} |\sin t|\, dt$$

$$= \frac{-.230}{m(m+1)^2} \cdot \frac{(m-1)m\pi}{4}.$$

As $m \geqq 5$, $(m-1)/(m+1) \geqq \frac{4}{6}$, and

$$h_{m2} < \frac{-.12}{m+1}.$$

Therefore

$$j_{m+1} - j_m = h_{m1} + h_{m2} < \frac{.92}{m+1};$$

$$j_m < j_5 + .92 \left(\frac{1}{6} + \frac{1}{7} + \cdots + \frac{1}{m} \right)$$

$$< j_5 + .92 \int_5^m \frac{du}{u} < 2.58 + .92 \log m - .92 \log 5$$

$$< 1.11 + .92 \log m.$$

We have then for $n \geqq 2$, $m \geqq 5$,

$$\rho_n < \frac{2}{\pi} \big(1.11 + .92 \log (2n+1)\big).$$

Since

$$2n + 1 \leqq \tfrac{5}{2} n,$$

when $n \geqq 2$, it follows that, for such values of n,

$$\rho_n < \frac{2}{\pi}(1.11 + .92 \log n + .92 \log \tfrac{5}{2}) < 1.25 + .59 \log n. \tag{20}$$

It is well known, though of course it does not follow from the work above, that ρ_n becomes infinite with n, and is in fact of the order of magnitude of $\log n$. Fejér showed that the values of ρ_n always increase with n, from a certain point on. He raised the question whether they do so from the very beginning.* It is not difficult to show that this is the case. It is an immediate consequence of (19) that $\rho_0 < \rho_1 < \rho_2$. In order to prove that $\rho_{n+1} > \rho_n$ in general, it will be sufficient to show that the value of the integral j_m increases with m, when $m \geqq 5$.

We write $j_{m+1} - j_m = h_{m1} + h_{m2}$ as before. It is obvious that, since $\sin[t/(m+1)]$ is less than unity,

$$h_{m1} > \frac{1}{m+1}.$$

With reference to h_{m2}, we have to review a considerable part of the work which led up to formula (14). In the interval $(0, \pi/2)$,

$$\cos x > 1 - \frac{x^2}{2},$$

$$x \cos x > x - .500\, x^3.$$

On the other hand,

$$\sin x < x - \frac{x^3}{6} + \frac{x^5}{120} < x - .166\, x^3 + .021\, x^3 = x - .145 x^3,$$

and hence

$$x \cos x - \sin x > - .355\, x^3,$$

$$\frac{d}{dx}\frac{\sin x}{x} > - .355\, x,$$

and, if $0 < x_1 < x_2 < \pi/2$,

$$\frac{\sin x_2}{x_2} - \frac{\sin x_1}{x_1} > - .355\, x_2 (x_2 - x_1).$$

Hence

$$m \sin\frac{t}{m} - (m+1)\sin\frac{t}{m+1} > t\left(-.355\frac{t}{m}\right)\left(\frac{t}{m} - \frac{t}{m+1}\right)$$

$$= - .355\frac{t^3}{m^2(m+1)}.$$

* FEJÉR, loc. cit., p. 31. See references in previous footnotes to the recent paper of GRONWALL.

From this and the second inequality in (12) it follows that

$$\left[(m+1)\sin\frac{t}{m+1}\right]^{-1}-\left[m\sin\frac{t}{m}\right]^{-1}>-.355\frac{\pi^2}{4}\frac{t}{m^2(m+1)}$$

$$>-.877\frac{t}{m^2(m+1)},$$

and so

$$h_{m2}>\frac{-.877}{m^2(m+1)}\int_0^{m\pi/2}t\,|\sin t|\,dt$$

$$>\frac{-.877}{m^2(m+1)}\frac{\pi}{2}(1+2+\cdots+m)\int_0^{\pi/2}\sin t\,dt$$

$$>\frac{-.69}{m}>\frac{-.83}{m+1},$$

the last inequality holding when $m \gtreqless 5$. We see thus that j_m does increase with m, and therefore ρ_n with n, from the beginning.

In connection with Theorem V, we were concerned directly with the value of $1+\rho_n$, for which we now have from (20) the inequality

$$1+\rho_n<2.25+.59\log n.$$

The relative importance of the first term of course decreases as n increases. If $n \gtreqless 5$, then $2.25<1.40\log n$, and $1+\rho_n<2\log n$; if $n \gtreqless 250$, then $2.25<.41\log n$, so that $1+\rho_n<\log n$. The asymptotic formula of Fejér mentioned above shows that

$$\lim_{n=\infty}\frac{1+\rho_n}{\log n}=\frac{4}{\pi^2}.$$

Of the various results which we are now in a position to formulate, along the line of Theorem V, the following may serve as an example:

THEOREM X.* *If $f(x)$ is a function of period 2π satisfying the Lipschitz condition* (1), *it is represented by the partial sum of its Fourier's series to terms of the nth order, provided $n \gtreqless 5$, with an error not exceeding* 6λ *$(\log n)/n$.*

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.

* This is an improvement over the result given in the article LEBESGUE II, p. 201 of the volume, 18 of the article; LEBESGUE was not concerned with the reduction of the constant in the numerator to its smallest value, nor with the determination of the point at which the inequality obtained begins to be true.